

UNIQUENESS OF CERTAIN SPHERICAL CODES

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1. Introduction. In this paper we show that there is essentially only one way of arranging 240 (resp. 196560) nonoverlapping unit spheres in \mathbf{R}^8 (resp. \mathbf{R}^{24}) so that they all touch another unit sphere, and only one way of arranging 56 (resp. 4600) spheres in \mathbf{R}^8 (resp. \mathbf{R}^{24}) so that they all touch two further, touching spheres. The following tight spherical t -designs are unique: the 5-design in Ω_7 , the 7-designs in Ω_8 and Ω_{23} , and the 11-design in Ω_{24} . It was shown in [20] that the maximum number of nonoverlapping unit spheres in \mathbf{R}^8 (resp. \mathbf{R}^{24}) that can touch another unit sphere is 240 (resp. 196560). Arrangements of spheres meeting these bounds can be obtained from the E_8 and Leech lattices, respectively. The present paper shows that these are the only arrangements meeting these bounds. In [2], [3], it was shown that there are no tight spherical t -designs for $t \geq 8$ except for the tight 11-design in Ω_{24} . The present paper shows that this and three other tight t -designs are also unique. There is already a considerable body of literature concerning the uniqueness of these lattices and their associated codes and groups ([5], [6], [8], [11], [13], [17]–[19], [21], [22], [27], [28]). However the results given here are believed to be new.

Our notation is that Ω_n denotes the unit sphere in \mathbf{R}^n and (\cdot, \cdot) is the usual inner product. An (n, M, s) spherical code is a subset C of Ω_n of size M such that $(\mathbf{u}, \mathbf{v}) \leq s$ for all $\mathbf{u}, \mathbf{v} \in C$, $\mathbf{u} \neq \mathbf{v}$.

Examples of spherical codes may be obtained from sphere packings ([15], [25]) via the following theorem, whose elementary proof is omitted.

THEOREM 1. *In a packing of unit spheres in \mathbf{R}^n let S_1, \dots, S_k be a set of spheres such that S_i touches S_j for all $i \neq j$. Suppose there are further spheres T_1, \dots, T_M each of which touches all the S_i . Then after rescaling the centers of T_1, \dots, T_M form an $(n - k + 1, M, 1/(k + 1))$ spherical code.*

Example 2. In the E_8 lattice packing in \mathbf{R}^8 there are 240 spheres touching each sphere, 56 that touch each pair of touching spheres, 27 that touch each triple of mutually touching spheres, and so on. From Theorem 1 the centers of these sets of spheres give rise to $(8, 240, 1/2)$, $(7, 56, 1/3)$, $(6, 27, 1/4)$, $(5, 16, 1/5)$, $(4, 10, 1/6)$ and $(3, 6, 1/7)$ spherical codes.

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Example 3. Similarly the Leech lattice in \mathbf{R}^{24} ([5], [14], [16], [26]) gives rise to (24, 196560, 1/2), (23, 4600, 1/3), (22, 891, 1/4), (21, 336, 1/5), (20, 170, 1/6) . . . spherical codes.

If C is an (n, M, s) spherical code and $\mathbf{u} \in C$ the *distance distribution* of C with respect to \mathbf{u} is the set of numbers $\{A_t(\mathbf{u}), -1 \leq t \leq 1\}$, where

$$A_t(\mathbf{u}) = |\{\mathbf{v} \in C: (\mathbf{u}, \mathbf{v}) = t\}|,$$

and the *distance distribution* of C is the set of numbers $\{A_t, -1 \leq t \leq 1\}$, where

$$A_t = \frac{1}{M} \sum_{\mathbf{u} \in C} A_t(\mathbf{u}).$$

Then the A_t satisfy

$$\begin{aligned} A_1 &= 1, \\ A_t &= 0 \quad \text{for } s < t < 1, \\ \sum_{-1 \leq t \leq s} A_t &= M - 1, \end{aligned}$$

and

$$\sum_{-1 \leq t \leq s} A_t P_k(t) \geq -P_k(1), \quad \text{for } k = 1, 2, 3, \dots,$$

where $P_k(x) = P_k^{(n-3)/2, (n-3)/2}(x)$ is a Jacobi polynomial in the notation of [1, Chapter 2]. For a proof of the last inequality see [9], [12], [16] or [20]. For a specified value of s an upper bound to M is therefore given by the following linear programming problem.

(P1) Choose $\{A_t, -1 \leq t \leq s\}$ so as to maximize

$$\sum_{-1 \leq t \leq s} A_t$$

subject to the inequalities

$$\begin{aligned} A_t &\geq 0, \\ (1) \quad \sum_{-1 \leq t \leq s} A_t P_k(t) &\geq -P_k(1), \quad \text{for } k = 1, 2, 3, \dots \end{aligned}$$

The dual problem may be stated as follows (compare the argument in [18, Chapter 17, § 4]).

(P2) Choose an integer N and a polynomial $f(t)$ of degree N , say

$$f(t) = \sum_{k=0}^N f_k P_k(t),$$

so as to minimize $f(1)/f_0$ subject to the inequalities

$$\begin{aligned} (2) \quad f_0 &> 0, f_k \geq 0 \quad \text{for } k = 1, 2, \dots, N, \\ (3) \quad f(t) &\leq 0 \quad \text{for } -1 \leq t \leq s. \end{aligned}$$

Since any feasible solution to the dual problem is an upper bound to the optimal solution of the primal problem, we have

$$(4) \quad M \leq f(1)/f_0$$

for any polynomial $f(t)$ satisfying (2) and (3).

2. Uniqueness of the code of size 240 in Ω_8 .

THEOREM 4 ([20]). *If C is an $(8, M, 1/2)$ code then $M \leq 240$.*

Proof. Consider the polynomial

$$\begin{aligned} f(t) &= \frac{320}{3} (t + 1) \left(t + \frac{1}{2}\right)^2 t^2 \left(t - \frac{1}{2}\right) \\ &= P_0 + \frac{16}{7} P_1 + \frac{200}{63} P_2 + \frac{832}{231} P_3 + \frac{1216}{429} P_4 + \frac{5120}{3003} P_4 \\ &\qquad\qquad\qquad + \frac{2560}{4641} P_6, \end{aligned}$$

where P_k stands for $P_k^{2 \cdot 5 \cdot 2 \cdot 5}(t)$. This satisfies (2) and (3) with $s = 1/2$, so from (4) we have $M \leq f(1)/f_0 = 240$.

THEOREM 5. *If (a) C is an $(8, 240, 1/2)$ code then (b) C is a tight spherical 7-design in Ω_8 (cf. [9], [10]), (c) C carries a 4-class association scheme (cf. [7], [26]), (d) the intersection numbers of this association scheme are uniquely determined, and (e) the distance distribution of C with respect to any $\mathbf{u} \in C$ is given by*

$$\begin{aligned} A_1(\mathbf{u}) &= A_{-1}(\mathbf{u}) = 1, \\ (6) \quad A_{1/2}(\mathbf{u}) &= A_{-1/2}(\mathbf{u}) = 56, \\ A_0(\mathbf{u}) &= 126. \end{aligned}$$

Proof. Let $\{A_t\}$ be the distance distribution of C . Then $\{A_t\}$ is an optimal solution to the primal problem (P1), and the polynomial $f(t)$ in (5) is an optimal solution to the dual problem (P2). The dual variables f_1, \dots, f_6 are nonzero, so by the theorem of complementary slackness [23] the primal constraints (1) must hold with equality for $k = 1, \dots, 6$.

The dual constraints (3) do not hold with equality except for $t = -1, \pm 1/2$ and 0 . Therefore the primal variables must vanish everywhere except perhaps for $A_{-1}, A_{\pm 1/2}$ and A_0 . From (1) these numbers satisfy the equations

$$(7) \quad A_{-1}P_k(-1) + A_{-1/2}P_k(-\frac{1}{2}) + A_0P_k(0) + A_{1/2}P_k(\frac{1}{2}) = -P_k(1),$$

for $k = 1, 2, \dots, 6$. Thus

$$(8) \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\frac{7}{2} & -\frac{7}{4} & 0 & \frac{7}{4} \\ \frac{63}{8} & \frac{9}{8} & -\frac{9}{8} & \frac{9}{8} \\ -\frac{231}{16} & \frac{33}{64} & 0 & -\frac{33}{64} \\ \frac{3003}{128} & -\frac{429}{256} & \frac{143}{128} & -\frac{429}{256} \\ -\frac{9009}{256} & \frac{1287}{1024} & 0 & -\frac{1287}{1024} \\ \frac{51051}{1024} & \frac{663}{2048} & -\frac{1105}{1024} & \frac{663}{2048} \end{bmatrix} \begin{bmatrix} A_{-1} \\ A_{-1/2} \\ A_0 \\ A_{1/2} \end{bmatrix} = \begin{bmatrix} 239 \\ -\frac{7}{2} \\ -\frac{63}{8} \\ -\frac{231}{16} \\ -\frac{3003}{128} \\ -\frac{9009}{256} \\ -\frac{51051}{1024} \end{bmatrix}$$

The unique solution is

$$(9) \quad A_{-1} = 1, A_{-1/2} = A_{1/2} = 56, A_0 = 126.$$

Since $A_{-1}(\mathbf{u}) \leq 1$ and $A_{-1} = 1$, we have $A_{-1}(\mathbf{u}) = 1$ for all $\mathbf{u} \in C$, and so the code is antipodal [9, p. 373]. Therefore (7) also holds for $k = 7$ and by [9, Theorem 5.5] C is a spherical 7-design. By [9, Definition 5.13] the design is tight, since $|C| = 2\binom{10}{3}$. By [9, Theorem 7.5] C carries a 4-class association scheme. Therefore $A_t(\mathbf{u}) = A_t$ is independent of \mathbf{u} for all t . This proves (b), (c) and (e). The numbers (9) are the valencies of the association scheme, and by [9, Theorem 7.4] determine all the intersection numbers. This proves (d).

THEOREM 6. *If condition (b) of Theorem 5 holds then so do (a), (c), (d) and (e).*

Proof. By definition $|C| = 2\binom{10}{3}$. From [9, Theorem 5.12] the inner products between the members of C are ± 1 and the zeros of

$$C_3(x) = 160(x + \frac{1}{2})x(x - \frac{1}{2}).$$

Thus all the A_t are zero except perhaps for $A_{\pm 1}$, $A_{\pm 1/2}$ and A_0 . From [9, Theorem 5.5] Eq. (7) holds for $k = 1, 2, \dots, 7$. The rest of the proof is the same as for Theorem 5.

In Example 2 we saw that the minimal vectors in the E_8 lattice form an $(8, 240, 1/2)$ code. Thus conditions (a)–(e) of Theorem 5 apply to this code. Conversely we have:

THEOREM 7. *If C is a tight spherical 7-design in Ω_8 there is an orthogonal transformation mapping C onto the minimal vectors of the E_8 lattice.*

Proof. From Theorem 6 the possible inner products in C are $0, \pm 1/2, \pm 1$. Let $C = \{\mathbf{u}_1, \dots, \mathbf{u}_{240}\}$ and let L be the lattice in \mathbf{R}^8 consisting of the vectors

$$\sum_{i=1}^{240} a_i \cdot \sqrt{2} \mathbf{u}_i, \quad a_i \in \mathbf{Z}.$$

Then L is an even integral lattice (cf. [19]). All such lattices have been classified (see [13], [19]), and are direct sums of the lattices $A_n (n \geq 1)$, $D_n (n \geq 4)$ and $E_n (n = 6, 7, 8)$. The only lattice of this type with at least 240 minimal vectors is E_8 , so L is isometric to E_8 and C is isometric to the minimal vectors in E_8 .

By combining Theorems 5 and 7 we obtain:

THEOREM 8. *There is a unique way (up to isometry) of arranging 240 nonoverlapping unit spheres in \mathbf{R}^8 so that they all touch another unit sphere.*

3. Uniqueness of the code of size 56 in Ω_7 .

THEOREM 9. *If C is a $(7, M, 1/3)$ code then $M \leq 56$.*

Proof. The proof here is parallel to the proof of Theorem 4, using the polynomial

$$f(t) = (t + 1)(t + 1/3)^2(t - 1/3).$$

THEOREM 10. *If (a) C is a $(7, 56, 1/3)$ code then (b) C is a tight spherical 5-design in Ω_7 , (c) C carries a 3-class association scheme, (d) the intersection numbers of this association scheme are uniquely determined, and (e) the distance distribution of C with respect to any $\mathbf{u} \in C$ is given by*

$$\begin{aligned} A_1(\mathbf{u}) &= A_{-1}(\mathbf{u}) = 1, \\ (10) \quad A_{1/3}(\mathbf{u}) &= A_{-1/3}(\mathbf{u}) = 27. \end{aligned}$$

Conversely (b) implies (a), (c), (d) and (e).

Proof. The proof is parallel to the proofs of Theorems 5 and 6.

For example the $(7, 56, 1/3)$ code given in Example 2 has properties (a)–(e). Conversely we have:

THEOREM 11. *If C is a tight spherical 5-design in Ω_7 there is an orthogonal transformation mapping C onto the $(7, 56, 1/3)$ code obtained from the E_8 lattice.*

Proof. Let C consist of the points $\mathbf{u}_1, \dots, \mathbf{u}_{56}$ lying on a unit sphere \mathbf{R}^7 centered at \mathbf{P} . Choose a point \mathbf{O} (in \mathbf{R}^8) so that $\sphericalangle \mathbf{u}_i \mathbf{O} \mathbf{P} = \pi/3$ for all i , and thus

$$\cos \sphericalangle \mathbf{u}_i \mathbf{O} \mathbf{u}_j = (1 + 3 \cos \sphericalangle \mathbf{u}_i \mathbf{P} \mathbf{u}_j)/4$$

for all i, j . Let \mathbf{v} be a unit vector along \mathbf{OP} (see Fig. 1). From Theorem 10 $\cos \angle \mathbf{u}_i \mathbf{P} \mathbf{u}_j$ takes the values ± 1 and $\pm 1/3$, so $\cos \angle \mathbf{u}_i \mathbf{O} \mathbf{u}_j$ takes the values 0, $\pm 1/2$ and 1. It follows that the vectors $\sqrt{3/2} \mathbf{O} \mathbf{u}_i$ ($1 \leq i \leq 56$) span an even integral lattice, containing at least $2(56 + 1) = 114$ minimal vectors (corresponding to $\pm C, \pm \mathbf{v}$). This lattice must therefore be either E_8 or $E_7 \oplus A_1$, and the latter is incompatible with (10).

By combining Theorems 10 and 11 we obtain:

THEOREM 12. *There is a unique way (up to isometry) of arranging 56 nonoverlapping unit spheres in \mathbf{R}^8 so that they all touch two further, touching, unit spheres.*

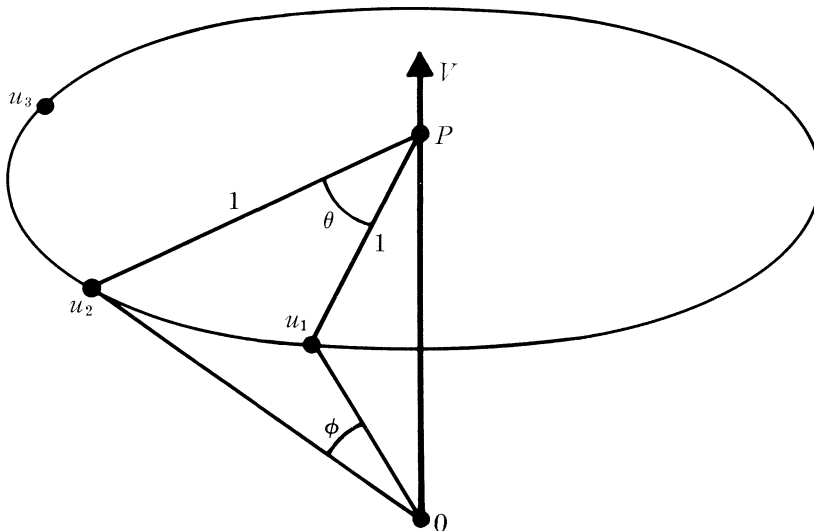


FIGURE 1. The construction used in the proof of Theorem 11: $\angle \mathbf{u}_i \mathbf{O} \mathbf{P} = \pi/3$ for all i , $|\mathbf{O} \mathbf{P}| = 1/\sqrt{3}$, $|\mathbf{O} \mathbf{u}_1| = |\mathbf{O} \mathbf{u}_2| = 2/\sqrt{3}$, and $\cos \phi = (1 + 3 \cos \theta)/4$

4. Uniqueness of the code of size 196560 in Ω_{24} .

THEOREM 13 ([20]). *If C is a $(24, M, 1/2)$ code then $M \leq 196560$.*

Proof. This parallels that of Theorem 4, using the polynomial

$$f(t) = (t + 1)(t + \frac{1}{2})^2(t + \frac{1}{4})^2 t^2 (t - \frac{1}{4})^2 (t - \frac{1}{2}).$$

THEOREM 14. *If (a) C is a $(24, 196560, 1/2)$ code then (b) C is a tight spherical 11-design in Ω_{24} , (c) C carries a 6-class association scheme, (d) the intersection numbers of this association scheme are uniquely determined, and*

(e) the distance distribution of C with respect to any $\mathbf{u} \in C$ is given by

$$(11) \quad \begin{aligned} A_1(\mathbf{u}) &= A_{-1}(\mathbf{u}) = 1, \\ A_{1/2}(\mathbf{u}) &= A_{-1/2}(\mathbf{u}) = 4600, \\ A_{1/4}(\mathbf{u}) &= A_{-1/4}(\mathbf{u}) = 47104, \\ A_0(\mathbf{u}) &= 93150. \end{aligned}$$

Conversely (b) implies (a), (c), (d) and (e).

Proof. The proof here is parallel to those of Theorems 5 and 6.

In Example 3 we saw that the minimal vectors in the Leech lattice when suitably scaled form a $(24, 196560, 1/2)$ code. We shall require an explicit description of this code, and take Λ to consist of the vectors

$$(\mathbf{0} + 2\mathbf{c} + 4\mathbf{x})/\sqrt{8}$$

and

$$(\mathbf{1} + 2\mathbf{c} + 4\mathbf{y})/\sqrt{8},$$

where $\mathbf{0} = 00 \dots 0, \mathbf{1} = 11 \dots 1, \mathbf{c}$ is any codeword in the binary Golay code g_{24} (cf. [18]) $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^{24}$, and $\sum x_i$ is even, $\sum y_i$ odd. The minimal vectors in Λ consist of

$$\begin{aligned} &759 \cdot 2^7 \text{ with components } ((\pm 2)^8 0^{16})/\sqrt{8}, \\ &2^2 \cdot \binom{24}{2} \text{ with components } ((\pm 4)^2 0^{22})/\sqrt{8}, \end{aligned}$$

$$(12) \quad 24 \cdot 2^{12} \text{ with components } ((\pm 1)^{23} (\mp 3)^1)/\sqrt{8}$$

and have norm $(x, x) = 4$.

This set of 196560 vectors will be denoted by Λ_4 . Then $\frac{1}{2}\Lambda_4$ is a $(24, 196560, 1/2)$ code to which conditions (a)-(e) of Theorem 14 apply. Conversely we have:

THEOREM 15. *If C is a tight spherical 11-design in Ω_{24} there is an orthogonal transformation mapping C onto $\frac{1}{2}\Lambda_4$.*

Proof. From Theorem 14 the distance distribution of C with respect to any $\mathbf{u} \in C$ is given by (11), and in particular the inner products in C are $0, \pm \frac{1}{4}, \pm \frac{1}{2}, \pm 1$. Let $C = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_{196560}\}$, and let L be the lattice in \mathbf{R}^{24} consisting of the vectors

$$\sum_{i=1}^{196560} a_i \cdot 2\mathbf{u}_i, \quad a_i \in \mathbf{Z}.$$

Then

$$(13) \quad (2\mathbf{u}_i, 2\mathbf{u}_j) \in \{0, \pm 1, \pm 2, \pm 4\}$$

and L is an even integral lattice. We shall establish Theorem 15 by showing that there is an orthogonal transformation mapping L onto 2Λ and C onto $\frac{1}{2}\Lambda_4$.

LEMMA 16. *The minimal norm (\mathbf{v}, \mathbf{v}) for $\mathbf{v} \in L, \mathbf{v} \neq \mathbf{0}$, is 4.*

Proof. The minimal norm is even, so suppose it is 2, with $(\mathbf{v}, \mathbf{v}) = 2, \mathbf{v} \in L$. For $\mathbf{u} \in 2C$ we have

$$|(\mathbf{u}, \mathbf{v})| = |\mathbf{u}| \cdot |\mathbf{v}| \cdot |\cos \sphericalangle(\mathbf{u}, \mathbf{v})| \leq 2\sqrt{2},$$

so $(\mathbf{u}, \mathbf{v}) \in \{0, \pm 1, \pm 2\}$ since L is integral. Suppose $(\mathbf{u}, \mathbf{v}) = 0$ for α choices of \mathbf{u} , $(\mathbf{u}, \mathbf{v}) = 1$ for β choices, and $(\mathbf{u}, \mathbf{v}) = 2$ for γ choices, with $\alpha + 2\beta + 2\gamma = 196560$. Without loss of generality we may assume $\mathbf{v} = (\sqrt{2}, 0, 0, \dots, 0)$.

Since C is an 11-design,

$$(14) \quad \frac{1}{196560} \sum_{i=1}^{196560} f(\mathbf{u}_i) = \frac{1}{\omega_{24}} \int_{\Omega_{24}} f(\xi) d\omega(\xi)$$

holds for any homogeneous polynomial $f(\xi_1, \xi_2, \dots, \xi_{24})$ of total degree ≤ 11 , where ω_{24} is the surface area of Ω_{24} [9, p. 372]. Let us choose $f = f_k = \xi_1^k$, for $k = 2$ and 4, so that

$$f_k(\mathbf{u}_i) = 2^{-k/2} ((\mathbf{u}_i, \mathbf{v}))^k.$$

The right hand side of (14) can be evaluated from

$$\begin{aligned} \frac{1}{\omega_{24}} \int_{\Omega_{24}} f_k(\xi) d\omega(\xi) &= \frac{1}{196560} \sum_{u \in 1/2\Lambda_4} f_k(\mathbf{u}) \\ &= \frac{8190}{196560} \quad \text{if } k = 2, \quad \text{or} \quad \frac{945}{196560} \quad \text{if } k = 4, \end{aligned}$$

using (12). The equations (14) now read

$$2\beta \cdot \frac{1^2}{8} + 2\gamma \cdot \frac{2^2}{8} = 8190,$$

$$2\beta \cdot \frac{1^4}{64} + 2\gamma \cdot \frac{2^4}{64} = 945,$$

which imply $\beta = 33600, \gamma = -210$, an impossibility.

LEMMA 17. *The set L_4 of vectors of norm 4 in L coincides with $2C$.*

Proof. By construction L_4 contains $2C$. Conversely take $\mathbf{u}, \mathbf{v} \in L_4$. Then $(\mathbf{u}, \mathbf{v}) \neq 3$, or else

$$(\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}) = (\mathbf{u}, \mathbf{u}) - 2(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v}) = 2,$$

contradicting Lemma 16. Similarly $(\mathbf{u}, \mathbf{v}) \neq -3$. Therefore $(\mathbf{u}, \mathbf{v}) \in \{0, \pm 1, \pm 2, \pm 4\}$ and $\sphericalangle(\mathbf{u}, \mathbf{v}) \geq \pi/3$ for $\mathbf{u} \neq \mathbf{v}$. From Theorem 13

$$|L_4| \leq 196560 = |2C|.$$

Therefore $L_4 = 2C$.

For $n \geq 3$ let D_n be the lattice in \mathbf{R}^n spanned by the vectors

$$(15) \quad \mathbf{g}_1 = \sqrt{2}(\mathbf{e}_1 + \mathbf{e}_2), \mathbf{g}_2 = \sqrt{2}(\mathbf{e}_1 - \mathbf{e}_2), \\ \mathbf{g}_3 = \sqrt{2}(\mathbf{e}_2 - \mathbf{e}_3), \dots, \mathbf{g}_n = \sqrt{2}(\mathbf{e}_{n-1} - \mathbf{e}_n),$$

with respect to an orthonormal basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ for \mathbf{R}^n ([4], [19]). There are $2n(n - 1)$ minimal vectors $((\pm\sqrt{2})^{20^{n-2}})$ in D_n . These lattices are nested: $D_3 \subseteq D_4 \subseteq \dots$.

LEMMA 18. (i) For any pair of vectors \mathbf{u}, \mathbf{v} in Λ_4 with $\angle(\mathbf{u}, \mathbf{v}) = \pi/2$ there are 44 vectors \mathbf{w} in Λ_4 with $\angle(\mathbf{u}, \mathbf{w}) = \angle(\mathbf{v}, \mathbf{w}) = \pi/3$. (ii) The same statement holds with Λ_4 replaced by $L_4 = 2C$. (iii) There are $2n - 4$ minimal vectors \mathbf{w} in D_n such that $\angle(\mathbf{g}_1, \mathbf{w}) = \angle(\mathbf{g}_2, \mathbf{w}) = \pi/3$.

Proof. (i) and (iii) are straightforward, and (ii) follows from (i) since Λ_4 and $2C$ are association schemes with the same parameters (Theorem 14).

LEMMA 19. L contains a sublattice isometric to D_3 .

Proof. For the generators $\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$ of D_3 we can take any triple $\mathbf{u}, \mathbf{v}, \mathbf{w} \in L_4$ with $\angle(u, v) = \pi/2, \angle(\mathbf{u}, \mathbf{w}) = \angle(\mathbf{v}, \mathbf{w}) = \pi/3$. Such a triple exists by Lemma 18(ii).

LEMMA 20. L contains a sublattice isometric to D_n , for $n = 3, 4, \dots, 24$.

Proof. We proceed by induction on n . Suppose the assertion holds for $n \geq 3$. By choosing a suitable orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ L_4 contains vectors $\mathbf{g}_1, \dots, \mathbf{g}_n$ given by (15) which span D_n . By Lemma 18 (ii) there are 44 vectors \mathbf{w} in L_4 with $\angle(\mathbf{g}_1, \mathbf{w}) = \angle(\mathbf{g}_2, \mathbf{w}) = \pi/3$. By Lemma 18 (iii) at least one of these is not a minimal vector of D_n . Then this vector \mathbf{w} is not in $\mathbf{R}D_n$. (For suppose $\mathbf{w} = w_1\mathbf{e}_1 + \dots + w_n\mathbf{e}_n$. Since $\angle(\mathbf{g}_1, \mathbf{w}) = \angle(\mathbf{g}_2, \mathbf{w}) = \pi/3, w_1 = \sqrt{2}$ and $w_2 = 0$. For $3 \leq i \leq n,$

$$\sqrt{2}(\mathbf{e}_1 \pm \mathbf{e}_i) \in L_4 \cap D_n \subseteq 2C,$$

and therefore

$$(\mathbf{w}, \sqrt{2}(\mathbf{e}_1 \pm \mathbf{e}_i)) \in \{0, \pm 1, \pm 2\}$$

from (13). This implies $w_3 = w_4 = \dots = w_n = 0$, and contradicts $(\mathbf{w}, \mathbf{w}) = 4$.) Choose \mathbf{e}_{n+1} so that $\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\}$ is an orthonormal basis for $\mathbf{R}\langle D_n, \mathbf{w} \rangle$, and suppose

$$\mathbf{w} = w_1\mathbf{e}_1 + \dots + w_n\mathbf{e}_n + w_{n+1}\mathbf{e}_{n+1}.$$

The above argument shows that $w_1 = \sqrt{2}, w_2 = \dots = w_n = 0$, and $w_{n+1} = \pm\sqrt{2}$. Therefore $\langle D_n, \mathbf{w} \rangle = D_{n+1} \subseteq L$.

LEMMA 21. L is isometric to Λ .

Proof. From Lemma 20 we may choose an orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_{24}$ so that $2C$ contains the vectors $(\pm\sqrt{2})^2 \mathbf{0}^{22}$. Let $\mathbf{u} = (u_1, \dots, u_{24})/\sqrt{8}$ be any vector in $2C$. From (13) the inner products of \mathbf{u} with the vectors $(\pm\sqrt{2})^2 \mathbf{0}^{22}$ are $0, \pm 1, \pm 2, \pm 4$. By considering the inner products with $(\sqrt{2}, \pm\sqrt{2}, 0, \dots, 0)$ we obtain

$$\begin{aligned} u_1^2 + u_2^2 + \dots + u_{24}^2 &= 32, \\ \frac{1}{2}(u_1 \pm u_2) &\in \{0, \pm 1, \pm 2, \pm 4\}, \\ u_1, u_2, \dots &\in \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}. \end{aligned}$$

Suppose $u_1 = \pm 5$. Then another u_i , say u_2 , is zero. The inner product of \mathbf{u} with $(\sqrt{2}, \sqrt{2}, 0, \dots, 0)$ is $5/2$, a contradiction. Proceeding in this way it is not difficult to show that the only possibilities for the components of \mathbf{u} are

$$((\pm 2)^{80^{16}})/\sqrt{8}, ((\pm 4)^{20^{22}})/\sqrt{8}, \text{ and } ((\pm 1)^{23}(\pm 3)^1)/\sqrt{8}.$$

In particular u_1, \dots, u_{24} are integers with the same parity.

It remains to show that these vectors are the same as those in Λ_4 (see (12)). To see this we define a binary linear code \mathcal{C} of length 24 by taking as codewords all binary vectors \mathbf{c} such that there is a vector $\mathbf{u} \in L$ with

$$\mathbf{u} = (\mathbf{0} + 2\mathbf{c} + 4\mathbf{x})/\sqrt{8}$$

for some $\mathbf{x} \in \mathbf{Z}^{24}$. Then as in [5, p. 139] it follows that $\text{wt}(\mathbf{c}) \geq 8$ for $\mathbf{c} \neq \mathbf{0}$, and that there are at most 759 codewords of weight 8. Therefore $|\mathcal{C}| \leq 2^{12}$ (see for example [18, Fig. 1, p. 674]). The argument on page 140 of [5] now shows that the only way that $2\mathcal{C}$ can contain 196560 vectors \mathbf{u} is for these vectors to coincide with the minimal vectors (12) in Λ_4 .

This completes the proof of Theorem 15. By combining Theorems 14 and 15 we obtain:

THEOREM 22. *There is a unique way (up to isometry) of arranging 196560 nonoverlapping unit spheres in \mathbf{R}^{24} so that they all touch another unit sphere.*

5. Uniqueness of the code of size 4600 in Ω_{23} .

THEOREM 23. *If C is a $(23, M, 1/3)$ code then $M \leq 4600$.*

Proof. Use $f(t) = (t + 1)(t + 1/3)^2 t^2 (t - 1/3)$.

THEOREM 24. *If (a) C is a $(23, 4600, 1/3)$ code then (b) C is a tight spherical 7-design in Ω_{23} , (c) C carries a 4-class association scheme, (d) the intersection numbers of this association scheme are uniquely determined,*

and (e) the distance distribution of C with respect to any $\mathbf{u} \in C$ is given by

$$\begin{aligned} A_1(\mathbf{u}) &= A_{-1}(\mathbf{u}) = 1, \\ A_{1/3}(\mathbf{u}) &= A_{-1/3}(\mathbf{u}) = 891, \\ A_0(\mathbf{u}) &= 2816. \end{aligned}$$

Conversely (b) implies (a), (c), (d) and (e).

For example the (23, 4600, 1/3) code given in Example 3 has properties (a)–(e). Conversely we have:

THEOREM 25. *If C is a tight spherical 7-design in Ω_{23} there is an orthogonal transformation mapping C onto the (23, 4600, 1/3) code obtained from the Leech lattice.*

Proof. As in the proof of Theorem 11 we embed $C = \{\mathbf{u}_1, \dots, \mathbf{u}_{4600}\}$ in \mathbf{R}^{24} , choosing $\mathbf{0}$ so that $\sphericalangle \mathbf{u}_i \mathbf{OP} = \pi/3$ for all i (cf. Fig. 1). Then

$$\cos \sphericalangle \mathbf{u}_i \mathbf{O} \mathbf{u}_j \in \left\{ -\frac{1}{2}, 0, \frac{1}{4}, \frac{1}{2}, 1 \right\}.$$

Let L be the even integral lattice in \mathbf{R}^{24} spanned by the vectors $\sqrt{3} \mathbf{O} \mathbf{u}_i$. For convenience we set $\mathbf{U}_i = \sqrt{3} \mathbf{O} \mathbf{u}_i$.

LEMMA 26. *The minimum norm (\mathbf{v}, \mathbf{v}) for $\mathbf{v} \in L$, $\mathbf{v} \neq \mathbf{0}$, is 4.*

Proof. Suppose $\mathbf{v} \in L$ with $(\mathbf{v}, \mathbf{v}) = 2$, and write $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$ with $\mathbf{v}' \parallel \mathbf{OP}$, $\mathbf{v}'' \perp \mathbf{OP}$, $|\mathbf{v}'| = y$, $|\mathbf{v}''| = \sqrt{2 - y^2}$, and $\mathbf{U}_i = \mathbf{U}_i' + \mathbf{U}_i''$ with $\mathbf{U}_i' \parallel \mathbf{OP}$, $\mathbf{U}_i'' \perp \mathbf{OP}$, $|\mathbf{U}_i'| = 1$, $|\mathbf{U}_i''| = \sqrt{3}$. Then

$$\begin{aligned} (\mathbf{U}_i, \mathbf{v}) &= (\mathbf{U}_i', \mathbf{v}') + (\mathbf{U}_i'', \mathbf{v}'') \in \{0, \pm 1, \pm 2\}, \\ \cos \sphericalangle (\mathbf{U}_i'', \mathbf{v}'') &\in \frac{\{0, \pm 1, \pm 2\} - y}{\sqrt{3}\sqrt{2 - y^2}}. \end{aligned}$$

Since C is a tight 7-design, the set $\{\cos \sphericalangle (\mathbf{U}_i'', \mathbf{v}'') : 1 \leq i \leq 4600\}$ is symmetric about 0. Therefore $y \in \{0, \pm \frac{1}{2}, \pm 1\}$. First suppose $y = 0$. Then

$$\cos \sphericalangle (\mathbf{U}_i'', \mathbf{v}'') \in \left\{ -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\}.$$

Let these values occur $\gamma, \beta, \alpha, \beta, \gamma$ times respectively. Then by evaluating the 0th, 2nd and 4th moments of C with respect to \mathbf{v}'' , as in the proof of Lemma 16, we obtain the equations

$$\begin{aligned} \alpha + 2\beta + 2\gamma &= 4600 \\ \beta/3 + 4\gamma/3 &= 200 \\ \beta/8 + 8\gamma/9 &= 24, \end{aligned}$$

which imply $\gamma = -14$, an impossibility. Similarly for the other values of y .

LEMMA 27. *L contains a sublattice isometric to D_n , for $n = 3, 4, \dots, 24$.*

Proof. This is similar to the proof of Lemma 20, starting from the fact

that if we take $\mathbf{u}_1, \mathbf{u}_2 \in C$ with $\angle \mathbf{u}_1 \mathbf{O} \mathbf{u}_2 = \pi/2$, there are 42 vectors $\mathbf{u}_i \in C$ with

$$\angle \mathbf{u}_1 \mathbf{O} \mathbf{u}_i = \angle \mathbf{u}_2 \mathbf{O} \mathbf{u}_i = \pi/3.$$

Furthermore the vector $\mathbf{v} = 2\mathbf{OP} \in L$ also satisfies

$$\angle \mathbf{u}_1 \mathbf{O} \mathbf{v} = \angle \mathbf{u}_2 \mathbf{O} \mathbf{v} = \pi/3.$$

LEMMA 28. *L is isometric to Λ , and C is isometric to the (23, 4600, 1/3) code obtained from the Leech lattice.*

Proof. Let L_4 denote the set of minimal vectors in L . From Lemma 27 we may assume that L_4 contains all the vectors $((\pm 4^2 \mathbf{0}^{22}))/\sqrt{8}$, and that $\mathbf{v} = 2\mathbf{OP}$ is $(440 \dots 0)/\sqrt{8}$. As in Lemma 21 it follows that the vectors in L_4 have the form $((\pm 2)^8 \mathbf{0}^{16})/\sqrt{8}$, $((\pm 4^2 \mathbf{0}^{22}))/\sqrt{8}$, and $((\pm 1)^{23} (\pm 3)^1)/\sqrt{8}$. Furthermore the vectors U_i begin $(22 \dots)/\sqrt{8}$, $(40 \dots)/\sqrt{8}$, $(04 \dots)/\sqrt{8}$, $(31 \dots)/\sqrt{8}$, or $(13 \dots)/\sqrt{8}$. The code \mathcal{C} is defined as in Lemma 21: it is a linear code of minimum distance 8 containing at most 2^{12} codewords. The zero codeword corresponds to the vectors U_i beginning $(40 \dots)/\sqrt{8}$ or $(04 \dots)/\sqrt{8}$, and there are at most $2 \cdot 2 \cdot 22$ of them. The codewords of weight 8 beginning $11 \dots$ correspond to the vectors U_i beginning $(22 \dots)/\sqrt{8}$. The number of such codewords is at most 77 ([18, Fig. 3, p. 688]), and there are at most $2^5 \cdot 77$ corresponding U_i . The remaining U_i come from codewords beginning $10 \dots$ or $01 \dots$, and there are at most $2 \cdot 2^{10}$ of them ([18, Fig. 1, p. 674]). Since $2 \cdot 2 \cdot 22 + 2^5 \cdot 77 + 2 \cdot 2^{10} = 4600$, all the inequalities in the argument must be exact. In particular the codewords of weight 8 beginning $11 \dots$ must form the unique Steiner system $S(3, 6, 22)$ (cf. [28]), and hence L must be the Leech lattice.

This completes the proof of Theorem 25. By combining Theorem 24 and 25 we obtain:

THEOREM 29. *There is a unique way (up to isometry) of arranging 4600 unit spheres in \mathbf{R}^{24} so that they all touch two further, touching, unit spheres.*

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